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breakdown of Antarctis in the Tertiary. Both Gondwana-Land and Antarctis had a far longer duration than any of the continents of today.

In the period immediately preceding the isolation of these continental masses they were united at the west; that is, in the occidental South Atlantic, the south polar land extended continuously into the land regions of the Gondwana Continent. This we know from the determinations of the Devonian strand lines in southern South America, the Falkland Islands and South Africa.

The Devonian of these latitudes is a unit both in life and in sedimentation. In this regard it is wholly unlike the Devonian of Eria, the east-west continent of the North, and it is a conclusion that is irrefragable on the basis of the intimate and refined analysis that such determinations require and have received. The haphazard observer may be blind to these radical distinctions, especially when basing interpretations upon a knowledge of the strand faunas of Eria. The known extent of these Southern Devonian shore faunas, as pointed out by the writer (Fosseis Devonianos do Parana; Monographias, Vol. I, Servico Geol. e Mineral do Brazil, 1913), indicates the union of the Gondwana and Antarctis continents throughout the Devonian. The extent of this Devonian land bridge across the Atlantic is clearly shown by the unity of shore faunas in South Africa, Sao Paulo, Argentine and Bolivia, and the indication is of a land composed of Paleozoic strata of still earlier date. This is Falklandia, the parent land asylum out of which, in Postcarboniferous time, western Gondwana and Antarctis were carved. The name is appropriately taken, for on the Falkland Islands the Devonian marine strata border the Gangamopteris (Glossopteris) beds of Gondwana-Land.

Other names which have been suggested for these pre-Gondwana austral lands have been founded on inadequate evidence. The "South Atlantic Island" of Frech indicated a Devonian land which had no connection with South Africa; Katzer imagined a north-south Devonian Atlantis running along the axis of the ocean, and Schwarz drew, with somewhat freer hand, his "Flabellites Land," as an undivided land mass along the Atlantic axis reaching from the north, and at the south spreading west and east to join Frech's "South Atlantic Island."

ON THE REAL FOLDS OF ABELIAN VARIETIES

By S. Lefschetz

Department of Mathematics, University of Kansas

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1. In this note we propose to find the number of real folds that an abelian variety of rank one may have and to establish some simple properties regarding their connectivity.

An abelian variety of genus p, V_p, belonging to an S_n, $(n > p)$, is defined by equations

$$x_j = f(u_1, u_2, \ldots u_n), \; (j = 1, 2, \ldots n) \tag{1}$$

where the (f)'s are $2p$-ply periodic meromorphic functions of the (u)'s, the periods forming a matrix Ω of a type called, after Scorza, a Riemann matrix. The rank of V_p is the number of distinct systems of values of the (u)'s, modulo Ω, corresponding to an arbitrary point of V_p. It is easy to show that to any Ω corresponds a V_p of rank one. Suppose indeed the (f)'s so chosen that any other periodic meromorphic function φ belonging to Ω be a rational function of them. If V_p were of rank > 1 then to every set of values $(u_1, u_2, \ldots u_p)$, would correspond another $(u_1', u_2', \ldots u_p')$, distinct modulo Ω, such that $\varphi(u) = \varphi(u')$ whatever φ. Choosing p such functions with non zero jacobian we find that the (u')'s are functions of the (u)'s. But $\varphi(u_1' - \alpha_1, \ldots u_p' - \alpha_p) = \varphi(u_1 - \alpha_1, \ldots u_p - \alpha_p)$ whatever the constants α, since the function at the right is of same type as φ. Hence at once $du_h' = du_h$, $u_h' - u_h = \beta_h$. The (β)'s are constants which obviously form a set of periods of the (f)'s, and therefore V_p is effectively of rank one.—All abelian varieties of rank one belonging to Ω are birationally equivalent.

2. Denoting by $\bar{m}$ the complex conjugate of any number m, a real variety of S_m is defined by the condition that when it contains one of the points, said to be conjugate, (x), $(\bar{x})$, it contains the other. Let then Ω be a Riemann matrix with a corresponding real V_p of rank one. Any simple integral of the first kind of V_p is of the form $u = \int \Sigma R_j \, dx_j$, where the (R)'s are rational in the (x)'s. Replacing in them all coefficients by their conjugates we obtain a new integral of the first kind of V_p, say (u), and u is a linear combination of the two integrals $u + (u), -i(u - (u))$, which are of real form. Hence V_p possess p independent integrals of real form, $u_1, u_2, \ldots u_p$. If V_p has a real point and we take it for lower limit of integration, our integrals will assume conjugate values at conjugate points of V_p.

Let now γ be any linear cycle of V_p and $\bar{\gamma}$ its transformed by T, transformation of the variety permuting each point with its conjugate. T transforms $\gamma + \bar{\gamma}$ into itself and $\gamma - \bar{\gamma}$ into its opposite. As $2\gamma, = (\gamma + \bar{\gamma}) + (\gamma - \bar{\gamma})$, the double of any cycle is the sum of two others transformed by T the one into itself and the other into its opposite. The periods of u_h with respect to cycles of the first type are real, and with respect to those of the other type they are pure complex. If q is the number of cycles of one type $2p - q$ is that of the cycles of the other type. As the real and complex parts of the periods of p independent integrals of the first kind with respect to $2p$ independent cycles form a non zero determinant, we must have $q = 2p - q$, or $q = p$. Finally we may single out $2p$ cycles $\gamma_1, \gamma_2, \ldots \gamma_{2p}$, such that $T.\gamma_h = \gamma_h$, $T.\gamma_{p+h} = -\gamma_{p+h}$; $h \leqq p$ while the cycles $m_1\gamma_1 + m_2\gamma_2 + \ldots + m_{2p}\gamma_{2p}$, ($m_k$ integer), include the double of any cycle of V_p. The corresponding period matrix for the (u)'s is of the type

$$\| \omega_{h,1}, \omega_{h,2}, \ldots . \omega_{h,p}; i\omega_{h,p+1}, i\omega_{h,p+2}, \ldots i\omega_{h,2p} \|, \qquad (2)$$
$$(h = 1, 2, \ldots 2p)$$

the (ω)'s being real.

Remark.—In all this V_p could be replaced by any real irreducible algebraic variety of irregularity p.

3. Conversely if to Ω corresponds a V_p of rank one with $2p$ linear cycles $\gamma_1, \gamma_2, \ldots \gamma_{2p}$, such that the cycles $m_1\gamma_1 + m_2\gamma_2 + \ldots . + m_{2p}\gamma_{2p}$ include the double of any other, while p independent integrals of the first kind $u_1, u_2, \ldots u_p$, have with respect to them a period matrix of type (2), V_p is birationally equivalent to a real abelian variety. Indeed as a consequence of the assumptions made, if the equations (1) represent V_p, and if $(\omega_1, \omega_2, \ldots \omega_p)$ are a set of periods of the (f)'s so are $(\bar{\omega}_1, \bar{\omega}_2, \ldots \bar{\omega}_p)$. Moreover the (f)'s are real meromorphic functions of the (u)'s and of a finite number of constants. If we replace these constants by their conjugates we obtain new functions $f(u_1, u_2, \ldots u_p)$ with the same periods as the (f)'s the equations

$$x_j' = f_j + \bar{f}_j, x_j'' = - i\,(f_j - \bar{f}_j), (j = 1, 2, \ldots n)$$

represent in an S_{2n} a real abelian variety birationally equivalent to V_p. This real abelian variety has ∞^{P} real points.

4. Assuming then V_p real with real folds, to determine their number we remark that at two conjuguate points the integrals u_h of No. 2 take conjuguate values $u_h' + iu_h'', u_h' - iu_h''$, modulo Ω. At the real folds the (u)'s are given by $u_h' + \frac{i}{2} \sum_1^p {}^{\mu}\, r_\mu\, \omega_{h,p+\mu}$, ($u_h'$ arbitrary, $r_\mu = 0$ or 1; $h = 1, 2, \ldots p$). There are in general $p - s$ periods of this type such that no linear combination of them with integral coefficients is of the form

$$\sum_1^p {}^{\mu} (m_\mu\, \omega_{h,\mu} + im_\mu'\, \omega_{h,p+\mu}), \; (h = 1, 2, \ldots p).$$

Hence $p - s$ of the integers r can be made equal to zero. Taking the others equal to zero or one yields 2^s distinct real folds.—That there are varieties having that number of real folds whatever $s \leqq p$ is shown by the canonical matrix

$$\left\| \begin{matrix} 1, 0 \ldots 0, a_{11}, a_{12}, \ldots a_{1p} \\ 0, 1 \ldots 0, a_{21}, a_{22}, \ldots a_{2p} \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ 0, 0 \ldots 1, a_{p1}, \ldots\ldots a_{pp} \end{matrix} \right\|$$

($a_{\mu\nu} = a_{\nu\mu} = \frac{1}{2}\, m_{\mu\nu} + i\, a_{\mu\nu}''$; $m_{\mu\nu}$ integers forming a matrix of rank s modulo 2; the quadratic form $\Sigma\, a_{\mu\nu}''\, x_\mu\, x_\nu$ is definite, positive.)

Hence a real abelian variety can have any number of real folds given by 2^s, $0 \leqq s \leqq p$.

5. Any real fold of the variety can be transformed into any other by a birational transformation belonging to the continuous group $(u_h, u_h + c_h)$. Hence

the real folds form equivalent p dimensional cycles, and the $q \leqq p$ dimensional ones which one of the folds contains are equivalent to some cycle, on any other. Moreover the real folds form homeomorphic manifolds. One of them is defined by the relations $u_h = \sum_{1}^{p} {}^{\mu} t_{\mu}\omega_{h\mu}$, where the (t)'s are real parameters varying from zero to one. It follows that any one of the folds corresponds point by point to the interior of a p-dimensional cube in the same manner as the points of a ring to those within a square. From this follows readily that the real folds are two sided manifolds of j-th index of connection equal to (p_j); this being also the number of independent real cycles of V_p. We may remark finally that each real fold is transformed into itself by the two sets of birational transformations, $(u_h, u_h + c_h)$, $(u_h, - u_h + c_h)$, where the (c)'s are real constants.

6. It is of interest to consider the types of real Jacobi varieties.—The Jacobi variety V_p of a curve of genus p, C_p, is an abelian variety of rank one belonging to the period matrix of p independent integrals of the first kind of C_p. Let C_p be real and its integrals of the first kind $v_1, v_2 \dots v_p$ real in form with a real point M_0 for lower limit of integration. We assume that C_p has $q > 0$ real branches. We may then take for V_p a representation (1), the (f)'s being real functions of the variables u, now defined by $u_h = \sum_{1}^{p} {}^{j} \int_{M_0}^{M_j} dv_h$, the points $M_1, M_2, \dots M_p$, being alone variable on C_p. In order that the point which corresponds to them on V_p be real it is necessary and sufficient that the set of the (M)'s coincide with that of their conjuguates,—a corollary of the fact that when a point of V_p moves on a real fold the increments of the (u)'s are real.

Let $B_1, B_2, \dots B_q$, be the real branches of C_p. If we constrain α_1 of the (M)'s to be on B_1, α_2 on $B_2, \dots \alpha_q$ on B_q, the corresponding point of V_p will be on a definite real fold F_1 of the variety, provided of course that $p - \Sigma\,\alpha_i$ be $\geqq 0$ and even. Assume one of the (α)'s, for example α_1, to be greater than one. I say that the fold F_2 corresponding to the integers $\alpha_1 - 2, \alpha_2, \dots \alpha_q$, coincides with F_1. Indeed a ready corollary from our previous discussion is that two distinct real folds of an abelian variety have no common points, while F_1 and F_2 intersect along the $p - 1$ dimensional manifold obtained when α_1 of the (M)'s of which two coincide are on B_1, the $p - \alpha_1$ remaining being disposed as previously. Hence if $q \leqq p + 1$, V_p has as many real folds as there are independent solutions of the congruence $\Sigma\alpha_i - p \equiv 0$, modulo 2, that is 2^{q-1}. If $q > p + 1$, V_p would have more than 2^p real folds. As this is impossible, $q \leqq p + 1$ and we have here a new proof of Harnack's theorem that a curve of genus p has at most $p + 1$ real branches.

We see then that a Jacobi variety can have any number of real folds given by 2^s, s being an integer at most equal to the genus p.